Nonlinear model for Marangoni convection

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We have constructed a Lorenz-like model for Marangoni convection with finite wave number in large aspect ratio situations. Within the model, there is exchange of stabilities at the onset of convection and beyond the onset there is onset of oscillations due to the presence of surface fluctuations. The oscillations become chaotic as the Marangoni number is increased.

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I. INTRODUCTION

For the problem of buoyancy driven convection, few mode models have been popular for studying the first effect of nonlinearities in the system. The first study of this kind was due to Lorenz [1], whose three mode truncation showed the presence of a strange attractor in the system and was the forerunner of all the subsequent work on chaos in dynamical systems. The latter studies on truncated modes have had a more restricted goal, generally studying the first effect of the nonlinearities on the convecting state. Of particular relevance in this context are the studies carried out on truncated systems for the onset of convection in the binary liquids [2–4] and on the onset of Kuepper-Lortz instability in convection in rotating fluids [5–7]. In the case of pure surface-tension driven convection [8–11] (Marangoni convection) a few mode truncation has not been attempted. This is presumably due to the technical difficulties associated with the free surface and the fact that the control parameter (Marangoni number) occurs only in a boundary condition. However, the free surface also offers interesting possibilities — there are waves, which can be propagated on a free surface. This could induce oscillatory convection. Oscillatory behavior has not yet been observed at the onset for large aspect ratio but at the same time it has not been possible to establish a principle of exchange of stabilities. The effect of free surface on Marangoni convection has been studied from the point of view of amplitude equations by Golovin *et al.* [12]. They have considered a situation where the long-wavelength instability and finite wavelength instabilities occur with almost the same Marangoni number and thus there is strong interaction between the modes with zero wave number and finite wave number. Coupled amplitude equations were written down for these modes and solved numerically by Kazhdan *et al.* [13]. We are working in a parameter regime ($Cr<0.000\,83$) where the long-wavelength instability is suppressed. Surface fluctuations will still exist since the top surface is free and our purpose will be to construct a dynamical system for a situation where a finite wave number instability occurs in the presence of surface deflections. The linear stability problem can be exactly solved analytically when the convection is driven purely by surface tension. For high surface tension, the onset occurs at finite wave numbers and it is known that the critical Marangoni number is $M_c=80.1$ and the critical wave number $a_c=2.0$. The velocity and temperature modes are also exactly known. This allows us to set up our Galerkin model with confidence. We will follow the strategy of using the exact velocity field at the onset as the lowest velocity mode and using a variational function for the temperature mode. The latter is pragmatic because we need an orthogonal set for introductory higher modes in the temperature field. How good is the variational function? This will be determined by how closely we can reproduce the onset Marangoni number from our truncated system. As we will see later, this occurs with an accuracy of 5%, thus establishing the reliability of our method.

The advantage of using the truncated Lorenz-like model is that one can now make predictions, which include the effect of the nonlinearity. The first thing that we find with our model is that at the onset, the convection can only be stationary. This is not a rigorous general result — it is only true within our model, but it is interesting that no experiment or computation has seen a Hopf bifurcation at the onset of Marangoni convection with large surface tension. The most important finding is that the steady convection bifurcates via Hopf bifurcation to an oscillatory pattern. Unlike in the Lorenz model for Rayleigh-Benard convection, this limit cycle is stable although over a small range. While this is a result of our model, this should have general validity — physically the limit cycle is stabilized by the surface fluctuations — the ripples on the surface due to surface tension. This is the most important point we wish to make — the secondary instability in this system should be an oscillatory instability. With a further increase in the Marangoni number, there is a transition to a chaotic state. There is a significant difference between the usual Lorenz attractor and the Marangoni attractor. Future experiments should be able to test the prediction.

The study of nonlinear effects has also been very restrictive because of the complexities associated with the boundary conditions. Consequently, we have undertaken the setting up of a dynamical system for Marangoni convection. Within the dynamical system found by us, there is an exchange of stabilities. However, the steady convection state does undergo a Hopf bifurcation, primarily because of the surface fluctuations. This feature should be generic to the system and experiments on Marangoni convection induced by heating from below in a large aspect ratio system should be able to see these oscillations if the region above the linear stability threshold is explored.

II. MATHEMATICAL MODEL

Before proceeding further, we should point out why a model, which can be considered primitive for the Rayleigh-

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Benard convection, is of our deep interest. This is due to the fact that even some of the very basic issues in Marangoni convection have not yet been perfectly settled. As mentioned briefly before, the principle of exchange of stabilities, which was proven more than half a century ago for the Rayleigh-Benard case, remain unproven for the Marangoni case. The fact that the truncated model does not allow a Hopf bifurcation of the conduction state is consequently a small step in the right direction. Experiments also have been few. Unlike the Rayleigh-Benard case (see, e.g., Ahlers [14]) the first experimental demonstration [15] of the correctness of the Pearson calculation was done only five years ego. In this situation, we believe that studying Lorenz models may be a good starting point. In fact Lorenz models can also be expanded to take into account more complicated problems. We will specialize, as is almost always the case with such studies, to two-dimensional geometry (the action is in the $x-z$ plane) and consider incompressible flow. A thin layer of fluid on a conducting plate is heated from below and ΔT is the temperature difference between the plate and the top layer of the fluid. The mean thickness of the fluid layer is "d." The conducting state has a linear temperature profile $T_c(z)=T_1-\beta z$, where T_1 is the temperature of the bottom plate and β is the temperature gradient. Our two primary field variables are the z component of the velocity $w(\vec{r},t)$ and the deviation θ of the temperature $T(\vec{r},t)$ from the conduction state profile. The x component u, of the velocity field can be found from the incompressibility condition.

We will be using dimensionless variables all through. All distances will be scaled by d, all time by d^2/ν, where ν is the kinematic viscosity, velocities by λ/d, where λ is the thermal diffusivity, and temperatures by ΔT. With the understanding that $\vec{\nabla}^2=\partial^2/\partial x^2+\partial^2/\partial z^2$, the equations describing Marangoni convection are

$$\vec{\nabla}^2\left(\nabla^2-\frac{\partial}{\partial t}\right)w=\{\vec{\nabla}\times\vec{\nabla}\times[(\vec{v}\cdot\vec{\nabla})\vec{v}]\}_z$$

$$=\frac{\partial}{\partial z}[\vec{\nabla}\cdot(\vec{v}\cdot\vec{\nabla})\vec{v}]-\nabla^2(\vec{v}\cdot\vec{\nabla})w, \qquad (1)$$

$$\left(\vec{\nabla}^2-\sigma\frac{\partial}{\partial t}\right)\theta=-w+(\vec{v}\cdot\vec{\nabla})\theta, \qquad (2)$$

where $\sigma=\nu/\lambda$ is the Prandtl number. The boundary conditions are given by

$$w=\frac{\partial w}{\partial z}=\theta=0 \quad \text{on} \quad z=0, \qquad (3)$$

$$w=\frac{\partial \eta}{\partial t} \quad \text{on} \quad z=1, \qquad (4)$$

$$\left[B-\frac{\partial^2}{\partial x^2}\right]\left[\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial x^2}\right)w-M\frac{\partial^2\theta}{\partial x^2}\right]$$

$$=M\cdot Cr\left[\frac{\partial^2}{\partial z^2}+3\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial t}\right]\frac{\partial w}{\partial z} \quad \text{on} \quad z=1, \qquad (5)$$

and

$$\frac{\partial\theta}{\partial z}=0 \quad \text{on} \quad z=1. \qquad (6)$$

In the above, the variable η denotes the surface fluctuation, $M=S_0\alpha\beta d^2/\rho\nu\lambda$ is the Marangoni number [$\alpha=(1/S_0)(\partial S_0/\partial T)$, S_0 being the mean surface tension], $Cr=\rho\nu\lambda/S_0 d$ is the Crispation number, and $B=\rho d^2 g/S_0$ is the Bond number, ρ being the density of the fluid.

Now our aim is to simplify the hydrodynamic equations by Galerkin trunaction. We will expand the independent variables (velocity and temperature) in a finite number of basis functions that will transform the nonlinear partial differential equations to a set of coupled nonlinear ordinary differential equations. Now comes the crucial issue of the choice of modes. The phenomenon we want to describe is the formation of stable cylindrical rolls. The axis of the cylinder is taken along the y direction. The selection of modes should be such that they reflect the circulating velocity and temperature fields and the transfer of heat from the lower plate to the upper plate. The boundary conditions [Eqs. (3)–(6)] should be matched by the functions of z for respective modes. We take two modes for the velocity w, which we write as

$$w=[A(t)g(z)+B(t)f(z)]\cos ax. \qquad (7)$$

If there is no surface tension in the upper surface of the fluid film, the velocity mode should be equal to that of the Lorenz model, i.e., then $w=A(t)g(z)\cos ax$. In Marangoni convection our intuitive prediction is that there should be another function of time, which will describe the time variation of the height of the upper surface due to the variation of surface tension over the surface of the film, i.e., $w=\partial\eta/\partial t$ on $z=1$. So the mode $A(t)$ corresponds to no surface fluctuation and the mode $B(t)$ contains all the information on the surface fluctuations. So, the requirement is $g(1)=0$, while $f(1)=1$. The function $g(z)$ is chosen in such a way that $w=A(t)g(z)\cos ax$ satisfies $\vec{\nabla}^4 w=0$ when $A(t)$ is a constant. This yields

$$g(z)=\sinh az-az\cosh az+\frac{ac-s}{s}z\sinh az, \qquad (8)$$

where $c=\cosh a$ and $s=\sinh a$. Similarly the function $f(z)$ also needs to satisfy $f(0)=df(0)/dz=0$ and accordingly, we choose

$$f(z)=z^2. \qquad (9)$$

With our choice of $g(z)$, we have ensured that the flow pattern is exactly obtained in the absence of surface fluctuations. The boundary conditions of θ make us choose it as

$$\theta=C(t)\sin\frac{\pi z}{2}\cos ax+D(t)\sin\frac{3\pi z}{2}. \qquad (10)$$

Here the choice of $\sin \pi z/2$ for the lowest mode is variational in character. How good this choice is will be determined by the critical Marangoni number of the model. We have to incorporate another mode because though the mode contain-

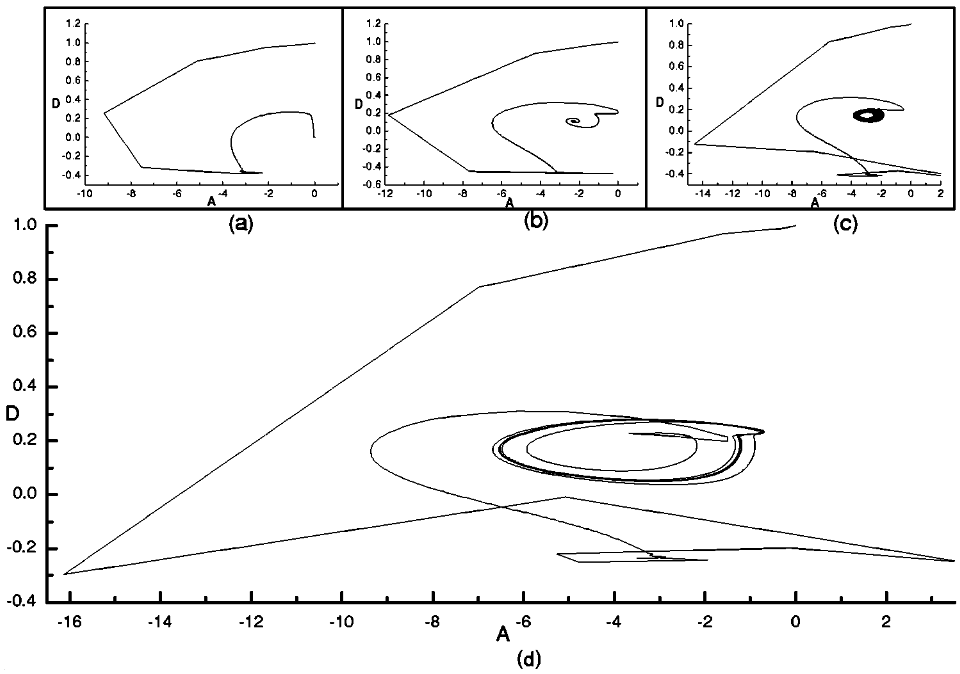

FIG. 1. (a) The conduction state at $M=75$; (b) the steady state at $M=150$; (c) the simple limit cycle at $M=202$; (d) the chaotic attractor at $M=280$.

ing $C(t)$ describes the rolls with finite wave number a, the averaging over a cell [i.e, integrating from $-(\lambda/2)$ to $\lambda/2$ along the x direction, where λ is the wavelength of the cell] gives zero value, meaning that there is no convection of heat from the lower plate to the upper plate. Now the lowest nontrivial mode that contributes to the convective heat transfer is $\sin 3\pi z/2$. From the incompressibility condition, the x component u of the velocity field works out to be $u=-(1/a)[A(t)g'(z)+2B(t)z]\sin ax$. To determine the dynamics of the modes, we insert w and θ from Eqs. (7) and (10) into Eq. (2) and match the corresponding Fourier coefficients. This gives equations for C and D. As expected these equations have nonlinear terms. For the modes A and B, we need to satisfy Eq. (1) in the mean and satisfy the boundary condition given in Eq. (5). Interestingly enough, satisfying Eq. (1) in the mean requires no contribution from the nonlinear term since the x dependent part of the contribution from this term is orthogonal to $\cos ax$. There is an unknown variable "a," the wave number. This is chosen, as in the Lorenz model, to be the critical wave number at the onset of stationary convection. This gives $a\simeq 2$. In the dynamics of A, B, C, and D all the coefficients are known. Approximating the coefficients we obtain the dynamical system

$$3\dot{A}+\frac{\dot{B}}{2}=-9B, \tag{11}$$

$$9\dot{A}-\dot{B}=-\left[\frac{13}{M\ Cr}-9\right]A+\left(20+\frac{3}{2MCr}\right)B+\frac{4}{Cr}C, \tag{12}$$

$$\sigma\dot{C}=\frac{A}{4}+\frac{3B}{4}-\frac{13C}{2}-AD-3BD, \tag{13}$$

$$\sigma\dot{D}=-\tfrac{45}{2}D+AC+3BC. \tag{14}$$

Thus we have constructed the model for the finite wave number Marangoni convection.

III. ANALYSIS

The trivial fixed point ($A=B=C=D=0$) corresponds to the conduction state. The destabilization of the trivial fixed point is found from the linear stability analysis of the fixed point, which yields the equations

$$3\,\delta\dot{A}+\tfrac{1}{2}\,\delta\dot{B}=-9\,\delta B, \tag{15}$$

$$9\,\delta\dot{A}-\delta\dot{B}=-\left[\frac{13}{MCr}-9\right]\delta A+\left(20+\frac{3}{2MCr}\right)\delta B+\frac{4}{Cr}\,\delta C, \tag{16}$$

$$\sigma\,\delta\dot{C}=\frac{\delta A}{4}+\frac{3\,\delta B}{4}-\frac{13}{2}\,\delta C. \tag{17}$$

The growth rate p of the fluctuation is obtained from the roots of a cubic equation. Exchange of stabilities $\partial/\partial t=0$ gives us

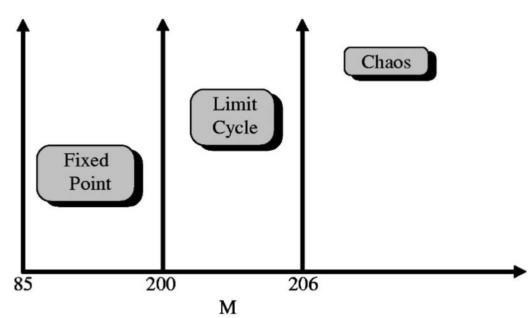

FIG. 2. Change of state with M.

$$M=M_0=\frac{169}{2+117Cr}. \tag{18}$$

Although we have a cubic equation for the growth rate — a Hopf bifurcation does not occur and this is the only possible bifurcation. For $Cr=0$ the critical Marangoni number is $M_0=\frac{169}{2}=84.5$, and the exact answer in the limit is $M_0=81$, which shows us how good the truncation is, at least for the study of the initial bifurcations. A conclusion can also be made within the model that the first bifurcation is always an exchange of stabilities. Oscillatory instability does not occur at the onset of convection. It has not been possible to establish the principle of exchange of stabilities within the complete set of Navier-Stokes' equations and the heat diffusion equation. However, the onset of convection is always stationary in all experiments with large aspect ratio. The oscillatory convection can occur only for small aspect ratios.

Now, for steady convection state, the nontrivial fixed point of Eqs. (11)–(14) is given by

$$A_0^2=\frac{169}{2}\frac{(1/M_0-1/M)}{13/M-9Cr}, \tag{19}$$

$$C_o^2=\frac{169}{32}\left(\frac{13}{M}-9Cr\right)\left(\frac{1}{M_0}-\frac{1}{M}\right), \tag{20}$$

$$D_0=\frac{169}{8}\left(\frac{1}{M_0}-\frac{1}{M}\right). \tag{21}$$

From the structure of the fixed point, it can be said immediately that the model is valid only for

$$M<\frac{13}{9}\frac{1}{Cr}. \tag{22}$$

From experimental data we know that typical values of Cr is 10^{-3}, and this makes the range of validity cover a large range of *M*. The nontrivial fixed point of Eqs. (19)–(21) exists for $M>M_0$ and it is easy to see that the bifurcation is forward. Our contention regarding the first bifurcation is supported by numerical integration of Eqs. (11)–(14). For $Cr=10^{-4}$, we show in Figs. 1(a) and 1(b), the trajectory settling down to the trivial fixed point for $M<85$ and to the nontrivial fixed point for $M>85$. Note that for $M=150$, the fixed point, where the trajectories end up in Fig. 1(b), ends up matching exactly with the nontrivial fixed point shown in Eqs. (19)–(21).

We now need to study the stability of the nontrivial fixed point. To do so, we write $A=A_0+\delta X$, $B=0+\delta Y$, $C=C_0+\delta Z$, and $D=D_0+\delta W$, insert in Eqs. (11)–(14) and linearize in δX, δY, δZ, and δW. The growth rate *p* satisfies a quartic equation. The condition of Hopf bifurcation leads to a quadratic for the critical Marangoni number and the lower branch is the one, which is relevant. For $Cr=10^{-4}$, this yields a critical Marangoni number $M_c\simeq 200$. The resulting limit cycle is stable and the result of numerical computation is shown in Fig. 1(c).

It is in these limit cycles that the role of the surface fluctuations become apparent. In the absence of the mode *B*, there are no stable limit cycles in the system. It is the coupling of the surface fluctuations to the heat diffusion that gives rise to the stable limit cycle. It is our contention that experiments on Marangoni convection, when carried on beyond the initial onset, will show onset of oscillations in large aspect ratio systems.

The limit cycle is stable over a small window. If *M* is raised beyond that, the chaotic state appears (Fig. 2). The interesting point about this attractor is its geometry. The strange attractor in the standard Lorenz case corresponds to the trajectory randomly switching from the neighborhood of unstable fixed point to another. Here the attractor is formed from the destabilization of a limit cycle and is essentially centered around an unstable fixed point.

In summary, we have carried out a Lorenz model study of the Marangoni convection occurring at a finite wave number. This supplements the study of Golovin *et al.* [12] for the long-wavelength convection, using an amplitude equation approach. Within our approach, we have managed to show that for the large aspect ratio system, the principle of stabilities is true. This result has not been proven in general for Marangoni convection. Based on our model, we have shown that cylindrical rolls will undergo a Hopf bifurcation as the first secondary instability. This feature is absent in the Rayleigh-Benard convection, where the oscillation has to do with the axis of the cylindrical rolls. Our model does not include modes giving three-dimensional structure, but it does make a prediction for a simple situation. This should provide motivation for experimental investigation of the secondary instabilities, which in the long run can inspire more complicated models.

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